Vietnam Journal of Computer Science
Vol. 7, No. 2 (2020) 1–19
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DOI: 10.1142/S2196888820500062

Problems of Optimal Control for a Class of Linear and Nonlinear Systems of the Economic Model of a Cluster

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Received 29 March 2019
Accepted 13 January 2020
Published 26 February 2020

For the mathematical model of a three-sector economic cluster, the problem of optimal control with fixed ends of trajectories is considered. An algorithm for solving the optimal control problem for a system with a quadratic functional is proposed. Control is defined on the basis of the principle of feedback. The problem is solved using the Lagrange multipliers of a special form, which makes it possible to find a synthesizing control. The problem of optimal stabilization for a class of nonlinear systems with coefficients that depend on the state of the control object is considered. The results obtained for nonlinear systems are used in the construction of control parameters for a three-sector economic cluster on an infinite time interval.

Keywords: Optimal control problem; three-sector economic cluster; Lagrange multiplier method; dynamical system; quadratic functional.

1. Introduction

The problem of optimal control for dynamical systems can be formulated as the problem of finding program control or constructing a synthesizing control that depends on the state of the system and the current time. In the first case, the problem can be solved using the Pontryagin[1,2] maximum principle. In the general case, the Pontryagin maximum principle gives the necessary conditions for optimality and allows one to obtain program control, depending on the current time. In the second

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case, Bellman's[3] dynamic programming method or Krotov[4] sufficient optimality conditions can be used.

In practice, there is a large number of optimal control problems (OCPs) for economic systems that are nonlinear systems with coefficients that depend on the state of the control object. In economic systems, it is required to achieve a certain level of economic development on a given planning horizon.

A three-sector (i.e. materials, labor resources, and production assets) economic model and the necessary conditions for an optimal balanced growth of the economy are given in Kolemaev.[5] Various aspects of the analysis of economic growth with the development of deterministic and stochastic three-sector dynamical models of open and closed types are presented in the works of Dzhusupov *et al.*,[6] De,[7] Dobrescu *et al.*,[8] Zhang,[9] Zhou and Xue,[10] Sen.[11]

The fundamental work of Aseev *et al.*[12] provides the foundations of the mathematical theory of optimal control of dynamical systems on an infinite interval using the Pontryagin maximum principle. As an example, a two-sector model of optimal economic growth with a random price jump is considered.

The paper of Shnurkov and Zasypko[13] studies the OCP for a dynamic three-sector model of the economy on the basis of the maximum principle. The OCP considered by them is a problem with free right ends of trajectories with scalar control, representing specific investments in the fund-creating sector of the economy. Note that in this paper a frequent case is considered, when the investment changes only in the fund-creating sector.

In contrast to the above works, we consider the OCP with fixed ends of trajectories, in a finite time interval. In this paper, we propose to use an approach based on sufficient optimality conditions using Lagrange multipliers of a special type, which allows us to represent the desired control in the form of synthesizing control, depending on the state of the system and the current time. In addition, this method makes it possible to take into account the existing restrictions on the values of controls. It should also be emphasized that this study considers the setting of the OCP for a three-sector economic model of a cluster in which the shares of labor and investment resources for all three sectors of the economy can simultaneously change.

Note that the peculiarity of the OCP considered in this paper is that the trajectories of the system must pass through given points at the initial and final instants of time (i.e. the left and right ends of the trajectories are fixed). The problem is considered on a finite time interval, there are restrictions on the values of controls, the task of constructing a synthesizing control is posed. In the work of Aipanov and Murzabekov,[14] the method of Lagrange multipliers is used to solve this problem, with the use of multipliers of a special form, which allows to obtain optimal control in the form of a control sum with feedback and program control.

The proposed approach is used to solve the problem of optimal distribution of investment and labor resources in a three-sector economic model of a cluster.

In control theory, much attention is paid to the problem of stability studies in nonlinear systems and to the problem of stabilizing nonlinear control systems.

Recently, new control algorithms for nonlinear systems have appeared, based on the use of Riccati equations with coefficients that depend on the state of the system. The ambiguity of the representation of a nonlinear system in the form of a system of linear structure and the absence of sufficiently universal algorithms for solving the Riccati equation whose parameters also depend on the state give rise to many possible suboptimal solutions. Due to the fact that there are many different types of nonlinearities in technical and economic systems, different approaches to constructing feedback control laws that are rational with respect to a given quality criterion arise (Afanas'ev[15] and Dmitriev[16]).

In practice, there is a large number of optimal control tasks for economic systems that are nonlinear systems with coefficients that depend on the state of the control object. In this paper, in contrast to the paper by Murzabekov *et al.*,[17] Sec. 3 is supplemented. The problem of optimal stabilization for a class of nonlinear systems with coefficients that depend on the state of the control object is considered. It is proposed to use an approach based on sufficient optimality conditions, which allows us to represent the desired control in the form of a stabilizing control, depending on the state of the nonlinear system and the current time. In addition, this method makes it possible to take into account the existing restrictions on the values of controls. The results obtained for nonlinear systems are used in the construction of control parameters for a three-sector economic cluster on an infinite time interval.

2. Statement of the OCP for a Three-Sector Economic Model of a Cluster

Consider a three-sector economic model of a cluster, described by a system of six differential and algebraic equations (Kolemaev[5])

$$\begin{aligned} \dot{k}_i &= -\lambda_i k_i + \frac{s_i}{\theta_i} x_1, \quad k_i(0) = k_i^0, \quad \lambda_i > 0 \quad (i = 0, 1, 2); \\ x_i &= \theta_i b_i k_i^{\alpha_i}, \quad b_i > 0, \quad 0 < \alpha_i < 1 \quad (i = 0, 1, 2), \end{aligned} \tag{1}$$

as well as three balance conditions:

$$s_0 + s_1 + s_2 = 1, \quad s_i \geq 0 \quad (i = 0, 1, 2), \tag{2}$$

$$\theta_0 + \theta_1 + \theta_2 = 1, \quad \theta_i \geq 0 \quad (i = 0, 1, 2), \tag{3}$$

$$(1 - \beta_0) x_0 = \beta_1 x_1 + \beta_2 x_2, \quad \beta_i \geq 0 \quad (i = 0, 1, 2). \tag{4}$$

Here, the state of the system is described by the vector (k_0, k_1, k_2) and $(s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$ is the vector of control. The initial state of the system is (k_0^0, k_1^0, k_2^0), where $k_i^0 = k_i(0)$ the capital-labor ratios of ith sector ($i = 0$ material, $i = 1$ creation of funds, and $i = 2$ production) at $t = 0$. We will consider the problem of transferring the system to the state (k_0^s, k_1^s, k_2^s) for the interval $[t_0, T]$. As the desired final state (k_0^s, k_1^s, k_2^s), we choose the equilibrium state of the system, which is determined by equating the right-hand sides of the differential equations (1) to zero, i.e.

$$k_0^s = \frac{s_0\theta_1 b_1 (k_1^s)^{\alpha_1}}{\lambda_0\theta_0}, \quad k_1^s = \left(\frac{s_1 b_1}{\lambda_1}\right)^{\frac{1}{1-\alpha_1}}, \quad k_2^s = \frac{s_2\theta_1 b_1 (k_1^s)^{\alpha_1}}{\lambda_2\theta_2}. \tag{5}$$

The values of capital-labor ratios $k_i^s, (i = 0, 1, 2)$ in the steady state (5) depend on the controls $(s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$, for which Murzabekov *et al.*[18] determined the values of $(s_0^s, s_1^s, s_2^s, \theta_0^s, \theta_1^s, \theta_2^s)$, solving the nonlinear programming in problem to maximize the specific consumption: $x_2 \to \max$.

In the state (k_0^s, k_1^s, k_2^s) (5), the right-hand sides of the differential equations (1) vanish, which means the constant in time of the values of capital-labor ratios $k_i^s, (i = 0, 1, 2)$ are constant in the equilibrium state.

Using three balance relations (2)–(4), a problem with six controls $(s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$ can be reduced to a task with three controls, denoted later through (s_1, θ_1, v_2), using the controls

$$s_0 = v_2(1 - s_1), \quad s_2 = (1 - v_2)(1 - s_1).$$

We write the system of differential equations (1) in deviations with respect to the equilibrium state of the system using the following notation:

$$\begin{aligned} &y_1 = k_0 - k_0^s, \quad y_2 = k_1 - k_1^s, \quad y_3 = k_2 - k_2^s, \\ &u_1 = s_1 - s_1^s, \quad u_2 = v_2 - v_2^s, \quad u_3 = \theta_1 - \theta_1^s, \\ &\dot{y}_i = f_i(y, u), \quad y_i(t_0) = y_i^0 \quad (i = 0, 1, 2), \end{aligned} \tag{6}$$

where $y = (y_1, y_2, y_3)^*$ denotes the state vector of the object, $u = (u_1, u_2, u_3)^*$ denotes the control vector. Linearizing the system (6), we obtain a vector differential equation of the form

$$\dot{y}(t) = Ay(t) + Bu(t), \quad t \in [t_0, T], \tag{7}$$

where the elements of the matrices $A = \|a_{ij}\|_{3\times 3}$ and $B = \|b_{ij}\|_{3\times 3}$ are determined by the formulas

$$a_{ij} = \frac{\partial f_i(y, u)}{\partial y_j}, \quad b_{ij} = \frac{\partial f_i(y, u)}{\partial u_j} \quad (i, j = 1, 2, 3), \tag{8}$$

with $y = (0, 0, 0)^*$ and $u = (0, 0, 0)^*$.

It should be noted that the controllability criteria for nonlinear systems of the form (6) were obtained in the works of Klamka,[19] and for discrete systems in the works of Klamka.[20] The system (7) is controllable, i.e. matrices and satisfy the controllability criterion defined in the works of Klamka.[19]

The initial and final states of the system are given as

$$y(t_0) = y_0, \quad y(T) = 0. \tag{9}$$

Note that the desired final state of the system $y(T) = 0$ is an equilibrium state in which the specific consumption is maximized and a balanced growth of the sectors of the economy is ensured.

The control vector components $u = (u_1, u_2, u_3)^*$ satisfy two-way constraints of the following type:

$$-s_1^s \leq u_1 \leq 1 - s_1^s, \quad -v_2^s \leq u_2 \leq 1 - v_2^s, \quad -\theta_1^s \leq u_3 \leq 1 - \theta_1^s, \tag{10}$$

which are derived from the source constraints $0 \leq s_1^s \leq 1$, $0 \leq v_2^s \leq 1$, $0 \leq \theta_1^s \leq 1$.

We consider the OCP: it is required to find the control $u(t)$, which takes the system (7) from the given initial state $y(t_0) = y_0$ to the equilibrium state $y(T) = 0$ for the interval $[t_0, T]$, while minimizing the target functional

$$J(u) = \frac{1}{2} \int_{t_0}^{T} \{y^*(t)Qy(t) + u^*(t)Ru(t)\}\, dt, \tag{11}$$

where Q and R are positive semidefinite and positive definite (3×3)-matrices, respectively.

Thus, we obtain the so-called LQ-problem (linear-quadratic OCP) (7)–(11), in which the ends of the trajectories of the system are fixed: $y(t_0) = y_0$, $y(T) = 0$, i.e. it is required to ensure the optimal path through specified start and end points. This part is considered in the works of Murzabekov *et al.*[17]

3. Statement Problems of Optimal Stabilization

We consider the optimal control task for a class of nonlinear systems with coefficients that depend on the state of the object. The problem of transferring a nonlinear system from a given initial state (k_0^0, k_1^0, k_2^0) to the desired state (k_0^s, k_1^s, k_2^s) for a time interval $[t_0, \infty)$ is considered. The equilibrium state of the system is chosen as the desired final state (k_0^s, k_1^s, k_2^s) as (5).

The values of the assets in the state $k_i^s, (i = 0, 1, 2)$ in the equilibrium state depend on the controls $(s_0, s_1, s_2, \theta_0, \theta_1, \theta_2)$, for which the values $(s_0^s, s_1^s, s_2^s, \theta_0^s, \theta_1^s, \theta_2^s)$, were determined in the work Murzabekov.[18]

The mathematical model of the control object is written in the form of a system of differential equations in vector form

$$\dot{y}(t) = Ay(t) + BD(y)u(t) + B(D(y) - D(k^s))v^s \quad y(t_0) = y_0, \quad t \in [t_0, \infty), \tag{12}$$

using the following notation:

$$y_1 = k_1 - k_1^s, \quad y_2 = k_2 - k_2^s, \quad y_3 = k_0 - k_0^s,$$

$$u_1 = s_1 - v_1^s, \quad u_2 = s_2\theta_1/\theta_2 - v_2^s, \quad u_3 = s_0\theta_1/\theta_0 - v_3^s,$$

$$v_1^s = s_1^s, \quad v_2^s = s_2^s\theta_1^s/\theta_2^s, \quad v_3^s = s_0^s\theta_1^s/\theta_0^s,$$

$$f_1(y_1) = (y_1 + k_1^s)^{\alpha_1}, \quad f_1(y_2) = (y_2 + k_2^s)^{\alpha_2}, \quad f_1(y_3) = (y_3 + k_0^s)^{\alpha_0},$$

$$Ak^s + BD(k^s)v^s = 0,$$

$$A = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_1 \end{pmatrix},$$

$$D(y) = \begin{pmatrix} (y_1 + k_1^s)^{\alpha_1} & 0 & 0 \\ 0 & (y_1 + k_1^s)^{\alpha_1} & 0 \\ 0 & 0 & (y_1 + k_1^s)^{\alpha_1} \end{pmatrix},$$

$$D(k^s) = \begin{pmatrix} (k_1^s)^{\alpha_1} & 0 & 0 \\ 0 & (k_1^s)^{\alpha_1} & 0 \\ 0 & 0 & (k_1^s)^{\alpha_1} \end{pmatrix},$$

where $y = (y_1, y_2, y_3)^*$ stands for the state vector of the object, $u = (u_1, u_2, u_3)^*$ stands for the control vector. The components of the control vector $u = (u_1, u_2, u_3)^*$ satisfy two-sided constraints of the following form:

$$u(t) \in U(t) = \{u | \gamma_1(t) \leq u(t) \leq \gamma_2(t), \quad t \in [t_0, \infty), \quad \gamma_1, \gamma_2 \in C[t_0, \infty)\}. \tag{13}$$

We assume that the system (12) is controllable. The matrices A, B satisfy the controllability condition, that is, the condition $\text{rang}[B, AB, \ldots, A^{n-1}B]$ is satisfied. Denote by $\Delta(t_0)$ the set of all admissible controls satisfying the condition $u(t) \in U(t), t \in [t_0, \infty)$, and the corresponding trajectories $y(t, u)$ of system (12).

Let a functional be defined on the set $\Delta(t_0)$, which depends on the control and state of the object

$$\begin{aligned} J(u) = \frac{1}{2} \int_{t_0}^{\infty} \{ & y^*(t)Qy(t) + (D(y)u(t) + (D(y) - D(k^s))v^s)^* R(D(y)u(t) \\ & + (D(y) - D(k^s))v^s)\} \, dt, \end{aligned} \tag{14}$$

where Q is a positive semidefinite matrix, and $R, D(y)$ are positive definite matrices.

Task is set. It is required to find a stabilizing control $u(y, t)$ that satisfies the two-sided constraints (13) and takes the system (12) from the given initial state $y(t_0) = y_0$ to the desired equilibrium state $y(\infty) = 0$ for the time interval $[t_0, \infty)$, while minimizing the functional (14).

For the optimal control task (12)–(14), a control $u(y, t)$ is searched for the equilibrium position in the closed system to be asymptotically stable in the Lyapunov sense and to construct a criterion such that the control constructed has some optimality. For this we used a method based on the use of Lagrange multipliers of a special form (Aipanov and Murzabekov[14]).

3.1. *Solution problems of optimal stabilization*

To solve the problem, we add the system of differential equations (12) with the factor $\lambda = Ky(t)$ to the expression for the functional (14), and also the following expression:

$$\lambda_1^*(t)[\gamma_1 - u(t)] + \lambda_2^*(t)[u(t) - \gamma_2],$$

where $\lambda_1(t) \geq 0$, $\lambda_2(t) \geq 0$. As a result, we obtain the following functional:

$$\begin{aligned} L(y,u) = \int_{t_0}^{\infty} \Big\{ &\frac{1}{2} y^*(t) Q y(t) + \frac{1}{2}(D(y)u(t) + (D(y) - D(k^s))v^s)^* R(D(y)u(t) \\ &+ (D(y) - D(k^s))v^s) + (Ky(t))^*(Ay(t) + BD(y)u(t) \\ &+ B(D(y) - D(k^s))v^s - \dot{y}) + \lambda_1^*(t)[\gamma_1 - u(t)] + \lambda_2^*(t)[u(t) - \gamma_2] \Big\} dt, \end{aligned} \tag{15}$$

where K is a symmetric positive definite matrix.

We introduce the following functions:

$$V(y,t) = \frac{1}{2} y^*(t) K y(t), \quad \frac{\partial V}{\partial y} = Ky(t), \tag{16}$$

$$\begin{aligned} M(y,u,t) = \; &\frac{1}{2} y^*(t) Q y(t) + \frac{1}{2}(D(y)u(t) + (D(y) - D(k^s))v^s)^* R(D(y)u(t) \\ &+ (D(y) - D(k^s))v^s) + (Ky(t))^*(Ay(t) + BD(y)u(t) \\ &+ B(D(y) - D(k^s))v^s) + \lambda_1^*(t)[\gamma_1 - u(t)] + \lambda_2^*(t)[u(t) - \gamma_2]. \end{aligned} \tag{17}$$

Then the following representation of the functional (15):

$$L(u,y) = V(y_0, t_0) + \int_{t_0}^{\infty} M(y,u,t) dt. \tag{18}$$

The desired control is determined from relation

$$D^*(y)R(D(y)u(t) + (D(y) - D(k^s))v^s) = -D^*(y)B^*Ky(t) + \lambda_1(t) - \lambda_2(t), \tag{19}$$

where the matrix K is determined from the algebraic Riccati matrix equation

$$KA + A^*K - KBR^{-1}B^*K + Q = 0. \tag{20}$$

Here, the following notation is used:

$$\begin{aligned} \lambda_1(y,t) &= D(y)^* R D(y) \max\{0, \gamma_1 - \omega(y,t)\} \geq 0, \\ \lambda_2(y,t) &= D(y)^* R D(y) \max\{0, \omega(y,t) - \gamma_2\} \geq 0, \\ \omega(y,t) &= -D^{-1}(y)(D(y) - D(k^s))v^s - D^{-1}(y)R^{-1}B^*Ky(t), \\ A_1 &= A - BR^{-1}B^*K, \quad B_1 = BR^{-1}B^*, \\ \varphi(y,t) &= D^{-1}(y)R^{-1}(D^{-1})^*[\lambda_1(y,t) - \lambda_2(y,t)]. \end{aligned} \tag{21}$$

We note that the choice of the factors $\lambda_1(t) \geq 0 \quad \lambda_2(t) \geq 0$ of the form (21) ensures that the conditions for complementary slackness

$$\lambda_1^*(t)[\gamma_1 - u(t)] = 0, \quad \lambda_2^*(t)[u(t) - \gamma_2] = 0$$

Suppose that there exists a solution of Eq. (20), then the differential equation defining the law of motion of the system can be represented in the following form:

$$\dot{y}(t) = A_1 y(t) + BD(y)\varphi(y,t), \quad y(t_0) = y_0. \tag{22}$$

The results established for the problem of optimal control are formulated in the form of the following assertion.

Theorem 3.1. *Let Q be a positive semidefinite matrix, and $R, D(y)$ be positive definite matrices in the interval $[t_0, \infty)$. Suppose that the system (12) is completely controllable at time t_0. Then for the optimality of the pair $(y(t), u(t))$ in the task (12)–(14), it suffices to satisfy the following conditions:*

(1) *the trajectory $y(t)$ satisfies the differential equation*

$$\dot{y}(t) = A_1 y(t) + BD(y)\varphi(y,t), \quad y(t_0) = y_0; \tag{23}$$

(2) *the control $u(t)$ is defined as follows:*

$$u(y,t) = \omega(y,t) + \varphi(y,t). \tag{24}$$

The matrix K is determined from the algebraic Riccati matrix equation (20), the vector-function $\varphi(y,t)$ is defined by formula (21) in such a way as to ensure the fulfillment of constraints on control (13).

3.2. *Algorithm of solving the task of optimal stabilization*

We describe an algorithm for solving the optimal control task (12)–(14), convenient for realization on a PC.

(1) Integrate the system of differential equations (20) to determine the K in the interval $[t_0, \infty)$.
(2) Integrate the system of differential equations (23) in the interval $[t_0, \infty)$ under the initial conditions $y(t_0) = y_0$. In the process of integrating the system (23) into print, it is necessary to plot the optimal trajectory $y(t)$ and the optimal control $u(t)$.
(3) Let the state of the system $y(t)$ and the optimal control $u(t)$ be found, then,

$$f_i(y_i) = (y_i + k_i^s)^{\alpha_i} \quad (i = 0, 1, 2),$$
$$v = \frac{\beta_1 b_1 f_1(y_1) + \beta_2 b_2 f_2(y_2)^{\frac{1-u_1-v_1^s}{u_2+v_2^s}}}{(1-\beta_0) b_0 f_3(y_3)^{\frac{1-u_1-v_1^s}{u_3+v_3^s}} + \beta_2 b_2 f_2(y_2)^{\frac{1-u_1-v_1^s}{u_2+v_2^s}}} \tag{25}$$

ensure that condition (4) is satisfied;

$$s_1 = u_1 + v_1^s, \quad s_2 = (1-v)(1-s_1), \quad s_0 = v(1-s_1) \tag{26}$$

ensure the fulfillment of condition (2);

$$\theta_1 = \frac{1}{1 + \frac{s_0}{u_3+v_3^s} + \frac{s_2}{u_2+v_2^s}}, \quad \theta_2 = \frac{(1-v)(1-s_1)\theta_1}{u_2+v_2^s}, \quad \theta_0 = \frac{v(1-s_1)\theta_1}{u_3+v_3^s} \tag{27}$$

ensure that condition (3) is satisfied.

Numerical calculations were performed on a computer with the following values of the parameters (Table 1).

3.3. *Numerical calculations*

Example. Numerical calculations were made on a computer at the values of the parameters of the economic model of the cluster, which are given in Table 1.

The optimal control task is solved for the values of the initial state of the system $y(t_0)$, which are given in the following form (28), where $t_0 = 0$:

$$y(t_0) = (-700, -300, 300)^*, \tag{28}$$

and the matrices Q, R, K have the form

$$Q = \begin{pmatrix} 16 \cdot 10^{-4} & 0 & 0 \\ 0 & 8 \cdot 10^{-4} & 0 \\ 0 & 0 & 8 \cdot 10^{-4} \end{pmatrix},$$

$$R = \begin{pmatrix} 200 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 200 \end{pmatrix},$$

$$K = \begin{pmatrix} 0.020 \cdot 10^{-2} & 0 & 0 \\ 0 & 0.0013 \cdot 10^{-2} & 0 \\ 0 & 0 & 0.0013 \cdot 10^{-2} \end{pmatrix}.$$

The results of the system state calculations are shown in Fig. 1. It can be seen from Fig. 2 that the optimal controls do not exceed the limits of the region U determined by the constraints.

For the example under consideration, these restrictions have the form

$$-0.3476 \le u_1 \le 0.4524, \quad -0.1024 \le u_2 \le 0.6976, \quad -0.0794 \le u_3 \le 0.7205. \tag{29}$$

Here, the control components lies on the inside of the boundary area. The optimal values of the system states at the final time instant for $T = 100$: $y_1(T) = -0.0112$; $y_2(T) = -0.0284$; $y_3(T) = -0.0284$, and the optimal values of the controls at the final time instant for $T = 100$: $u_1(T) = -0.3118 \cdot 10^{-4}$; $u_2(T) = -0.5244 \cdot 10^{-4}$; $u_3(T) = -0.5126 \cdot 10^{-4}$.

Using the formulas (25)–(27), the optimal distribution of labor $(\theta_0(t),\ \theta_1(t), \theta_2(t))$ and investment $(s_0(t), s_1(t), s_2(t))$ resources is determined. Figures 3 and 4 shows the resource changes that satisfy the balance ratios (2)–(4). Significant investment

Table 1. Parameter values for a three-sector cluster economic model.

i	α_i	β_i	λ_i	b_i	s_i^*	θ_i^*	k_i^*
0	0.46	0.39	0.05	6.19	0.2763	0.3944	966.4430
1	0.68	0.29	0.05	1.35	0.4476	0.2562	2410.1455
2	0.49	0.52	0.05	2.71	0.2761	0.3494	1090.1238

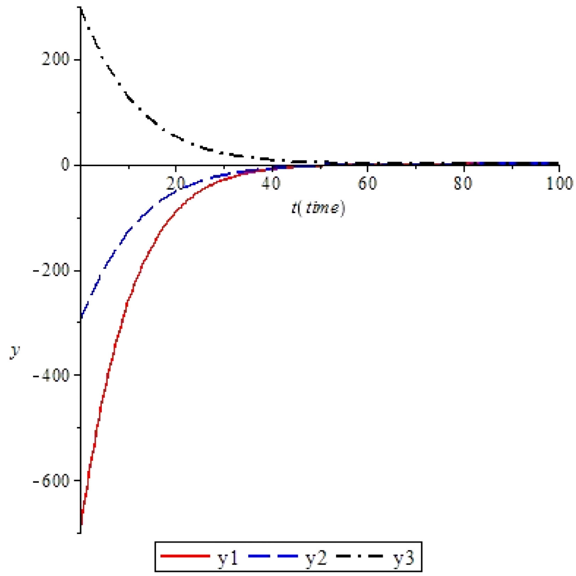

Fig. 1. Graphs of optimal trajectories $y(t)$.

$(s_0(t), s_1(t), s_2(t))$ and labor resources $(\theta_0(t), \theta_1(t), \theta_2(t))$ tend to $T = 100$ a steady state, with an estimate of approximation $|s_1(T) - s_1^s| = 0.3118 \cdot 10^{-4}$; $|s_2(T) - s_2^s| = 0.8870 \cdot 10^{-4}$; $|s_0(T) - s_0^s| = 0.120 \cdot 10^{-3}$; $|\theta_1(T) - \theta_1^s| = 0.9436 \cdot 10^{-5}$; $|\theta_2(T) - \theta_2^s| = 0.3460 \cdot 10^{-4}$; $|\theta_0(T) - \theta_0^s| = 0.4404 \cdot 10^{-4}$.

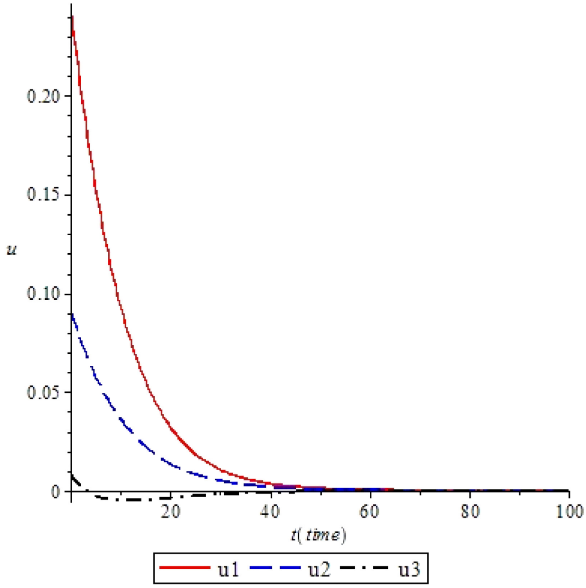

Fig. 2. Graphs of optimal controls $u(t)$.

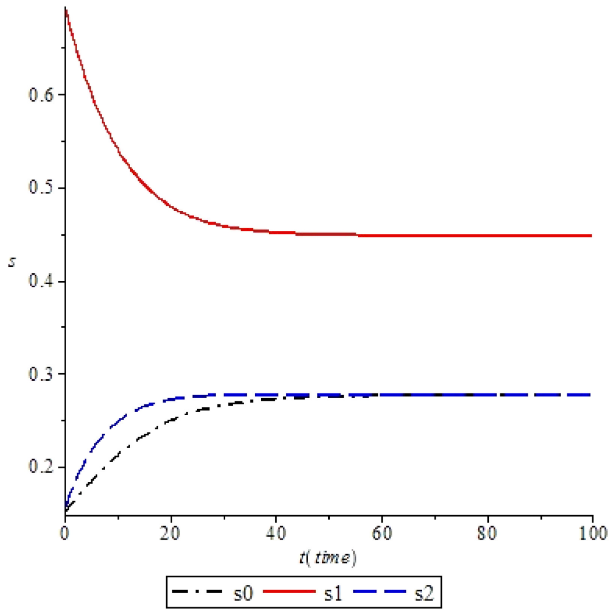

Fig. 3. Graphs of the optimal distribution of investment resources.

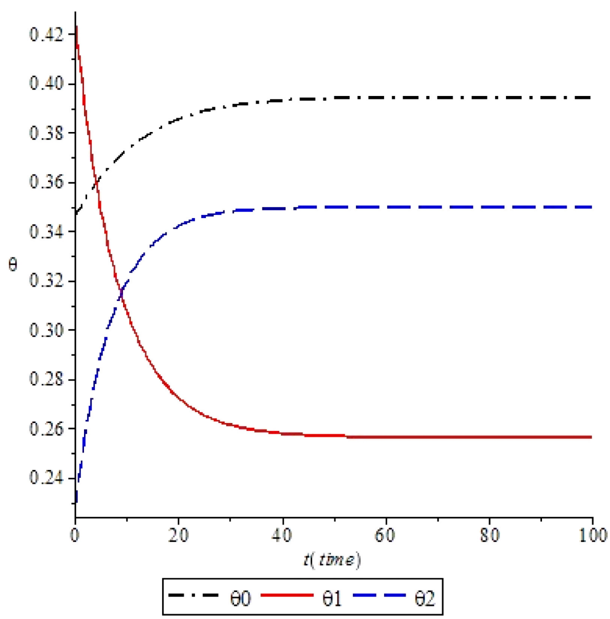

Fig. 4. Graphs of the optimal distribution of labor resources.

4. Solving a Problem with Limited Control

The mathematical model of the control object is written in the form of a system of differential equations in vector form

$$\dot{y}(t) = Ay(t) + BD(y)u(t) + B(D(y) - D(k^s))v^s \quad y(t_0) = y_0, \quad t \in [t_0, T], \tag{30}$$

using the following notation for Sec. 3.

The components of the control vector $u = (u_1, u_2, u_3)^*$ satisfy two-sided constraints of the following form:

$$u(t) \in U(t) = \{u | \gamma_1(t) \le u(t) \le \gamma_2(t), \quad t \in [t_0, T], \quad \gamma_1, \gamma_2 \in C[t_0, T]\}. \tag{31}$$

Let a functional be defined on the set $\Delta(t_0, T)$, which depends on the control and state of the object

$$\begin{aligned} J(u) = \frac{1}{2}\int_{t_0}^{T} &\{y^*(t)Q(y)y(t) + (u(t) + (E - D^{-1}(y)D(k^s))v^s)^* R(u(t) \\ &+ (E - D^{-1}(y)D(k^s))v^s)\}\, dt + \frac{1}{2}y^*(T)Fy(T), \end{aligned} \tag{32}$$

where $Q(y) = KBD(y)R^{-1}D^*(y)B^*K - KBD(k^s)R^{-1}D^*(k^s)B^*K + Q_1$ is a positive semidefinite matrix, and $R, F, D(y)$ are positive definite matrices.

Task is set. It is required to find a synthesizing control $u(y, t)$ that satisfies the two-sided constraints (31) and takes the system (30) from the given initial state $y(t_0) = y_0$ to the desired equilibrium state $y(T) = 0$ for the time interval $[t_0, T]$, while minimizing the functional (32).

To solve the problem, we add to the expression for functional (32) a system of differential equations (30) with a factor $\lambda = Ky + q(t)$, as well as the following expression:

$$\lambda_1^*[\gamma_1 - u(t)] + \lambda_2^*[u(t) - \gamma_2] + \lambda_3^*[y(t) - W(t, T)q(t)], \tag{33}$$

where $\lambda_1(t) \ge 0,\ \lambda_2(t) \ge 0$. As a result, we obtain the following functional:

$$\begin{aligned} L(y, u) = \int_{t_0}^{T} &\left\{\frac{1}{2}y^*(t)Q(y)y(t) + \frac{1}{2}(u(t) + (E - D^{-1}(y)D(k^s))v^s)^* R(u(t)\right. \\ &+ (E - D^{-1}(y)D(k^s))v^s) + (Ky + q(t))^*(Ay(t) + BD(y)u(t) \\ &+ B(D(y) - D(k^s))v^s - \dot{y}) + \lambda_1^*(t)[\gamma_1 - u(t)] + \lambda_2^*(t)[u(t) - \gamma_2] \\ &\left. + \lambda_3^*(t)[y(t) - W(t, T)q(t)]\right\} dt + \frac{1}{2}y^*(T)Fy(T), \end{aligned} \tag{34}$$

where $q(t)$ — vector of dimension $(n \times 1)$; K — symmetric positive definite matrix of dimension $(n \times n)$.

For the problem under consideration, the principle of liberation from connections is as follows: the initial OCP with constraints reduces to another problem, but without restrictions. Wherein, a new problem is formulated so that its solution would be a solution to the original problem (Murzabekov[21]).

We introduce in consideration the following functions:

$$V(y,t) = \frac{1}{2}y^* K y + y^* q(t), \quad \frac{\partial V}{\partial y} = Ky + q(t), \tag{35}$$

$$\begin{aligned} M(y,u,t) = {} & \frac{1}{2}y^*(t)Q(y)y(t) + \frac{1}{2}(u(t) + (E - D^{-1}(y)D(k^s))v^s)^* R(u(t) \\ & - (E - D^{-1}(y)D(k^s))v^s) + (Ky + q(t))^*(Ay(t) + BD(y)u(t) \\ & + B(D(y) - D(k^s))v^s) + y^*\dot{q}(t) + \lambda_1^*(t)[\gamma_1 - u(t)] \\ & + \lambda_2^*(t)[u(t) - \gamma_2] + \lambda_3^*(t)[y(t) - W(t,T)q(t)]. \end{aligned} \tag{36}$$

Then it's rightly to the following representation of functional (34):

$$L(u,y) = V(y_0,t_0) + \int_{t_0}^{T} M(y,u,t)dt - V(y(T),T) + \frac{1}{2}y^*(T)Fy(T). \tag{37}$$

The desired control is determined from the ratio

$$R(u(t) + (E - D^{-1}(y)D(k^s))v^s) = -D^*(y)B^*(Ky + q(t)) + (\lambda_1 - \lambda_2), \tag{38}$$

where the matrices are $K, W(t,T)$ and vector $q(t)$ satisfies the differential equations in the interval $t \in [t_0, T]$

$$KA + A^*K - KBD(k^s)R^{-1}B^*D^*(k^s)K + Q_1 = 0, \tag{39}$$

$$\dot{W}(t) = W(t)A_1(y) + A_1^*(y)W(t) - B_1(y), \quad W(T,T) = (F-K)^{-1}, \tag{40}$$

$$\dot{q}(t) = -A_1^*(y)q(t) + W^{-1}(t,T)BD(y)\varphi(y,t), \quad q(T) = (F-K)y(t). \tag{41}$$

Here, the following notation is used:

$$\begin{aligned} \lambda_1(y,t) &= R\max\{0, \gamma_1 - \omega(y,t)\} \geq 0, \\ \lambda_2(y,t) &= R\max\{0, \omega(y,t) - \gamma_2\} \geq 0, \\ \omega(y,t) &= -(E - D^{-1}(y)D(k^s))v^s - R^{-1}(t)B^*(Ky + q(t)), \\ A_1(y) &= A - B_1(y)K, \quad B_1(y) = BD(y)R^{-1}(t)D^*(y)B^*, \\ \varphi(y,t) &= R^{-1}[\lambda_1(y,t) - \lambda_2(y,t)]. \end{aligned} \tag{42}$$

Let there exist solutions of Eqs. (39) and (40), then the differential equations that determine the law of motion of the system can be represented in the following form:

$$\dot{y}(t) = A_1(y)y(t) - BD(y)R^{-1}D^*(y)B^*q(t) + BD(y)\varphi(y,t), \quad y(t_0) = y_0. \tag{43}$$

Note that the initial condition for differential equation (41) is determined from the following relation:

$$y(t) = W(t,T)q(t), \quad t \in [t_0, T]. \tag{44}$$

The results established for the OCP (30)–(32) we are formulated as the following statement.

Theorem 4.1. *Let $Q(y)$ be a positive semidefinite matrix, and R, F, $D(y)$ positively definite matrices in the interval $[t_0, T]$; the matrix $W_0 = W(t_0, T)$ is positive definite. Assume that system (30) is completely controllable at the moment of time t_0. Then for the optimality of the pair $(y(t), u(t))$ in the task (30)−(32), it suffices to satisfy the following conditions*:

(1) *the trajectory $y(t)$ satisfies the differential equation*

$$\dot{y}(t) = A_1(y)y(t) - BD(y)R^{-1}D^*(y)B^*q(t) + BD(y)\varphi(y,t), \quad y(t_0) = y_0. \tag{45}$$

(2) *the control $u(t)$ is defined as follows*:

$$u(y,t) = -(E - D^{-1}(y)D(k^s))v_s - R^{-1}D^*(y)B^*(Ky + q(t)) + \varphi(y,t). \tag{46}$$

Matrices $K, W(t,T)$ satisfies the differential equations (39) and (40), the function $q(t)$ satisfies differential equation (41), the vector-function $\varphi(y,t)$ is defined by formula (42) in such a way as to ensure the fulfillment of constraints on control (46).

4.1. *The algorithm for solving the problem*

We describe an algorithm for solving the optimal control task (30)–(32), convenient for realization on a PC.

(1) Integrate the system of differential equations (39) and (40) to determine the K and $W(t_0, T)$ in the interval $[t_0, T]$ with the condition $W(T,T) = (F - K)^{-1}$.
(2) Set the conditions $y(t_0) = y_0$, and calculate $q(t_0) = W^{-1}(t_0, T)y(t_0)$.
(3) Integrate the system of differential equations (43), (41) in the interval $[t_0, T]$ under the initial conditions $y(t_0) = y_0,\ q(t_0) = W^{-1}(t_0, T)y(t_0)$. In the process of integrating the system (43) and (41) into print, it is necessary to plot the optimal trajectory $y(t)$ and the optimal control $u(t)$.
(4) Let the state of the system $y(t)$ and the optimal control $u(t)$ be found, then,

$$f_i(y_i) = (y_i + k_i^s)^{\alpha_i} \quad (i = 0, 1, 2),$$
$$v = \frac{\beta_1 b_1 f_1(y_1) + \beta_2 b_2 f_2(y_2)\,\frac{1-u_1-v_1^s}{u_2+v_2^s}}{(1-\beta_0)b_0 f_3(y_3)\,\frac{1-u_1-v_1^s}{u_3+v_3^s} + \beta_2 b_2 f_2(y_2)\,\frac{1-u_1-v_1^s}{u_2+v_2^s}} \tag{47}$$

ensure that condition (4) is satisfied;

$$s_1 = u_1 + v_1^s, \quad s_2 = (1-v)(1-s_1), \quad s_0 = v(1-s_1) \tag{48}$$

ensure the fulfillment of condition (2);

$$\theta_1 = \frac{1}{1 + \frac{s_0}{u_3+v_3^s} + \frac{s_2}{u_2+v_2^s}}, \quad \theta_2 = \frac{(1-v)(1-s_1)\theta_1}{u_2+v_2^s}, \quad \theta_0 = \frac{v(1-s_1)\theta_1}{u_3+v_3^s}, \tag{49}$$

ensure that condition (3) is satisfied.

Numerical calculations were performed on a computer with the following values of the parameters (Table 1).

4.2. *Numerical calculations to determine the optimal distribution of labor and investment resources*

Example. Numerical calculations were made on a computer at the values of the parameters of the economic model of the cluster, which are given in Table 1.

The optimal control task is solved for the values of the initial state of the system $y(t_0)$, which are given in the following form (30), where $t_0 = 0$:

$$y(t_0) = (-700, -300, 300)^*, \tag{50}$$

and the matrices Q_1, R, K have the form

$$Q_1 = \begin{pmatrix} 16 \cdot 10^{-4} & 0 & 0 \\ 0 & 8 \cdot 10^{-4} & 0 \\ 0 & 0 & 8 \cdot 10^{-4} \end{pmatrix},$$

$$R = \begin{pmatrix} 200 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 200 \end{pmatrix},$$

$$K = \begin{pmatrix} 0.020 \cdot 10^{-2} & 0 & 0 \\ 0 & 0.0013 \cdot 10^{-2} & 0 \\ 0 & 0 & 0.0013 \cdot 10^{-2} \end{pmatrix}.$$

The results of the system state calculations are shown in Fig. 5. It can be seen from Fig. 6 that the optimal controls do not exceed the limits of the region U determined by the constraints.

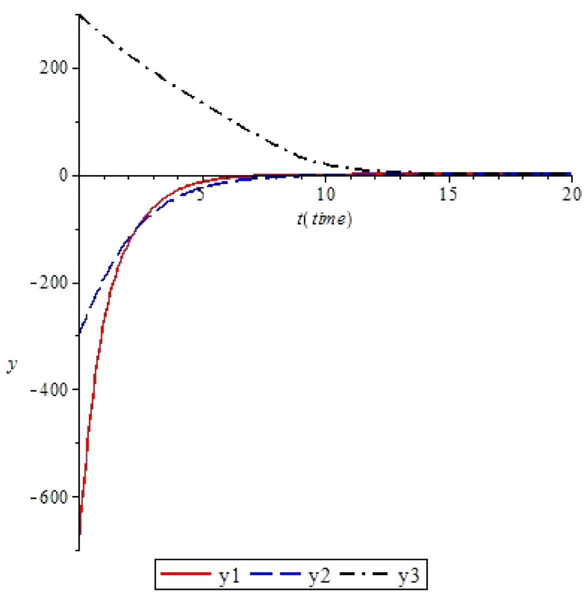

Fig. 5. Graphs of optimal trajectories $y(t)$.

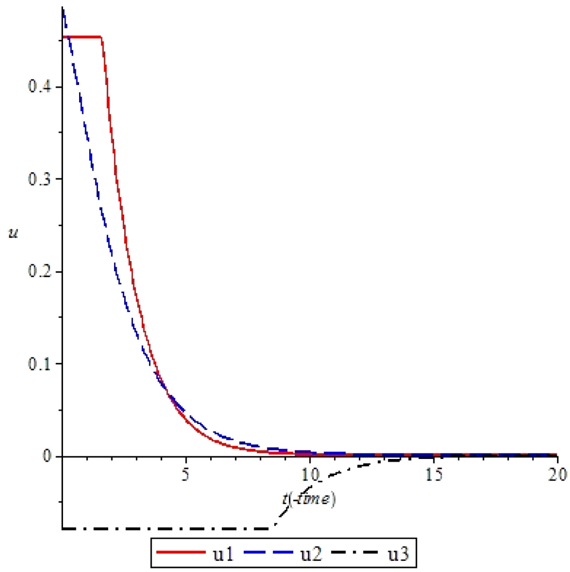

Fig. 6. Graphs of optimal controls $u(t)$.

For the example under consideration, these restrictions have the form

$$-0.3476 \leq u_1 \leq 0.4524, \quad -0.1024 \leq u_2 \leq 0.6976, \quad -0.0794 \leq u_3 \leq 0.7205. \quad (51)$$

Here, the control components $u_1(t)$ and $u_3(t)$ lie on the boundary of the region U in the time interval $[0, t_1]$ and $[0, t_2]$, respectively, then for $t \in [t_1, T]$, $t \in [t_2, T]$ go

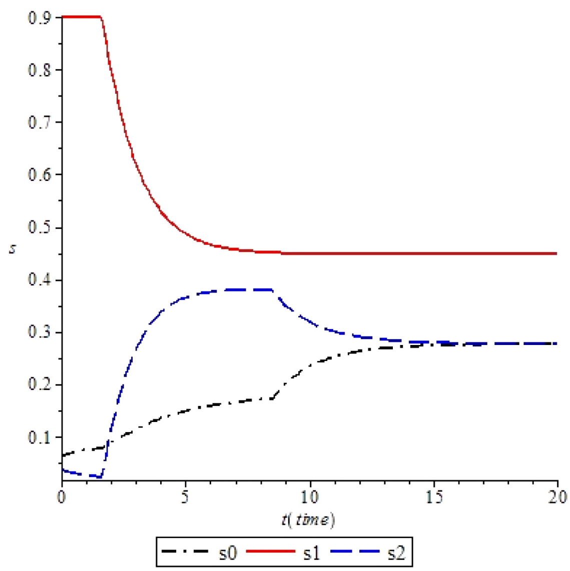

Fig. 7. Graphs of the optimal distribution of investment resources.

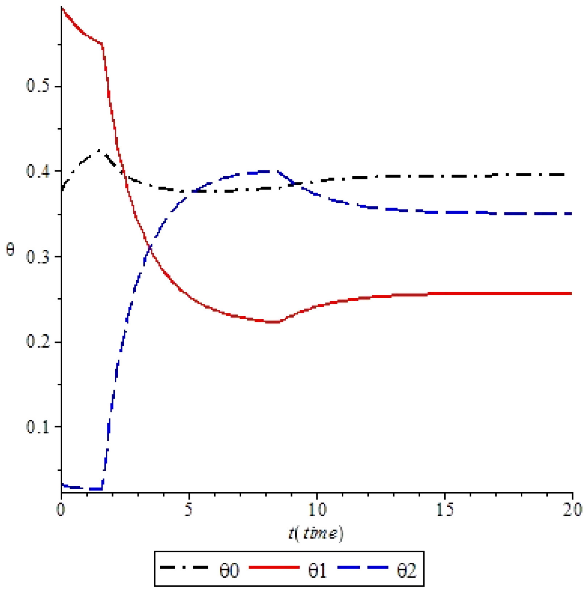

Fig. 8. Graphs of the optimal distribution of labor resources.

into the interior of the U domain. The control switching occurs at time $t_1 = 1.523$ for the component $u_1(t)$, and for $u_3(t)$ at $t_2 = 8.551$.

The optimal values of the system states at the final time instant for $T = 20$: $y_1(T) = -0.7320 \cdot 10^{-4}$; $y_2(T) = -0.2734 \cdot 10^{-2}$; $y_3(T) = 0.3146 \cdot 10^{-4}$, and the optimal values of the controls at the final time instant for $T = 20$: $u_1(T) = 6.5260 \cdot 10^{-7}$; $u_2(T) = 0.2432 \cdot 10^{-4}$; $u_3(T) = -0.2798 \cdot 10^{-4}$.

Using the formulas (36)–(38), the optimal distribution of labor $(\theta_0(t), \theta_1(t), \theta_2(t))$ and investment $(s_0(t), s_1(t), s_2(t))$ resources is determined. Figures 7 and 8 shows the resource changes that satisfy the balance ratios (2)–(4). Significant investment $(s_0(t), s_1(t), s_2(t))$ and labor resources $(\theta_0(t), \theta_1(t), \theta_2(t))$ tend to $T = 20$ a steady state, with an estimate of approximation $|s_1(T) - s_1^s| = 0.6526 \cdot 10^{-6}$; $|s_2(T) - s_2^s| = 0.3445 \cdot 10^{-3}$; $|s_0(T) - s_0^s| = 0.3451 \cdot 10^{-3}$; $|\theta_1(T) - \theta_1^s| = 0.1014 \cdot 10^{-3}$; $|\theta_2(T) - \theta_2^s| = 0.1730 \cdot 10^{-3}$; $|\theta_0(T) - \theta_0^s| = 0.7163 \cdot 10^{-4}$.

5. Conclusion

A new approach to the definition of synthesizing control based on the feedback principle is proposed. In Sec. 3, an algorithm for solving the optimal stabilization problem is developed and a nonlinear control is found using the matrix Riccati equation. In Sec. 4, a nonlinear synthesizing control is defined that brings the dynamic system to the required state in a finite time in the presence of control restrictions. The problems were solved using Lagrange multipliers depending on phase coordinates and time. Optimal controls are constructed on the basis of the feedback principle, and then control parameters (47)–(49) are selected.

The results obtained for nonlinear systems are used in the construction of control parameters for a mathematical model of a three-sector economic cluster. For this example, the optimal distribution of labor and investment resources that satisfy the balance ratios was determined. Figures 1–8 show the optimal trajectories and controls that satisfy the given constraints. The given example illustrates the use of the proposed method for controlling a nonlinear system.

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